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On pages 239–250 of the current volume of these Proceedings, in a paper entitled "Note on the chief theorem of Lie's theory of continuous groups," I pointed out an error in Lie's demonstration of the first fundamental theorem of his theory. In what follows I indicate how this error may be avoided and the demonstration completed.

Lie's error in the demonstration of the first fundamental theorem consists in neglecting conditions imposed at the outset upon certain auxiliary quantities $\mu_1, \mu_2, \ldots$ introduced in the course of the demonstration. Thus in the Continuierliche Gruppen, pp. 372–376 (and substantially in Transformationsgruppen, vol. III., pp. 558–564) Lie proceeds as follows: —

Being given a family with an ∞^r of transformations T_a, defined by the equations

$$x'_i = f_i\,(x_1 \ldots x_n,\ a_1 \ldots a_r) \quad (i = 1, 2 \ldots n),$$

containing the identical transformation, and, moreover, such that the x''s satisfy a certain system of differential equations, he defines by the introduction of new parameters μ a family of transformations E_μ,

$$x'_i = F_i\,(\overline{x}'_1 \ldots \overline{x}'_n,\ \mu_1 \ldots \mu_r) \quad (i = 1, 2 \ldots n),$$

each of which is generated by an infinitesimal transformation; Lie then establishes the symbolic equation

$$T_{\bar{a}}\, E_\mu = T_a,^*$$

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* If the equations defining the families of transformations T_a and E_μ are respectively,

$$x'_i = f_i\ (x_1 \ldots x_n,\ a_1 \ldots a_r) \quad (i = 1, 2 \ldots n),$$

and

$$x'_i = F_i(\overline{x}'_1 \ldots \overline{x}'_n,\ \mu_1 \ldots \mu_r) \quad (i = 1, 2 \ldots n),$$

the symbolic equation $T_{\bar{a}}\, E_\mu = T_a$ is equivalent to the simultaneous system of equations

where the a's and μ's are arbitrary, and

$$a_k = \Phi_k\,(\mu_1 \ldots \mu_r, \bar{a}_1, \ldots \bar{a}_r) \quad (k = 1, 2 \ldots r),$$

the Φ's being independent functions of the μ's.

For $\bar{a}_k = a_k^{(0)}$ $(k = 1, 2 \ldots r)$, the transformation $T_{\bar{a}}$ becomes the identical transformation; and therefore we have

$$E_\mu = T_{a^{(0)}}\, E_\mu = T_a,^*$$

where

$$a_k = \Phi_k\,(\mu_1 \ldots \mu_r, a_1^{(0)} \ldots a_r^{(0)}) \quad (k = 1, 2 \ldots r).$$

Thus every transformation of the family E_μ is a transformation of the family T_a. If, conversely, every transformation T_a belonged to the family E_μ, it would follow that

$$T_{\bar{a}}\; T_a = T_{a},†$$

that is to say, we should have shown that the family of transformations T_a forms a group.

But, although the Φ's are independent functions of the μ's, nevertheless the μ's in certain cases become infinite for certain systems of values of the a's; and infinite values of the μ's, by their definition, are excluded at the outset. ‡ We cannot then assume that every transformation T_a belongs to the family E_μ.

We may, however, proceed as follows: — For all values of the a's for which the functions

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$$\bar{x}'_i = f_i(x_1 \ldots x_n, \bar{a}_1 \ldots \bar{a}_r),$$
$$x'_i = F_i(\bar{x}'_1 \ldots \bar{x}'_n, \mu_1 \ldots \mu_r), \quad (i = 1, 2 \ldots n)$$
$$x'_i = f_i(x_1 \ldots x_n, a_1 \ldots a_r),$$

or to the functional equations

$$F_i(f_1(x, \bar{a}) \ldots f_n(x, \bar{a}), \mu_1 \ldots \mu_r) = f_i(x_1 \ldots x_n, a_1 \ldots a_r) \quad (i = 1, 2 \ldots n).$$

* That is,

$$F_i(\bar{x}'_1 \ldots \bar{x}_n, \mu_1 \ldots \mu_r) = F_i(f_1(x, a^{(0)}) \ldots f_n(x, a^{(0)}), \mu_1 \ldots \mu_r) = f_i(x_1 \ldots x_n, a_1 \ldots a_r)$$
$$(i = 1, 2 \ldots n),$$

since

$$\bar{x}'_i = f_i(x_1 \ldots x_n, a_1^{(0)} \ldots a_r^{(0)}) \quad (i = 1, 2 \ldots n).$$

† That is,

$$f_i(f_1(x, \bar{a}) \ldots f_n(x, \bar{a}), a_1 \ldots a_r) = f_i(x_1 \ldots x_n, a_1 \ldots a_r) \quad (i = 1, 2 \ldots n).$$

‡ These Proceedings, p. 217.

$\mu_j = M_j\,(a_1 \ldots a_r,\, a_1^{(0)} \ldots a_r^{(0)})$ $(j = 1, 2 \ldots r)$ are finite, we have

$$T_{\bar{a}}\, T_a = T_{\bar{a}}\, E_\mu = \mathrm{T}_a,$$

that is,

$$f_i\,(f_1\,(x, \bar{a}) \ldots f_n\,(x, \bar{a}),\, a_1 \ldots a_r) =$$
$$F_i\,(f_1\,(x, \bar{a}) \ldots f_n\,(x, \bar{a}),\, \mu_1 \ldots \mu_r) = f_i\,(x_1 \ldots x_n,\, a_1 \ldots a_r)$$
$$(i = 1, 2 \ldots n).$$

Let $\beta_1, \beta_2 \ldots$ be a system of values of the a's for which one, or more, of the corresponding μ's is infinite in all branches. Also let $b_1, b_2 \ldots$ be the system of values assumed by the a's for $a_k = \beta_k$ $(k = 1, 2 \ldots r)$.

Since the functions f are continuous functions of the variables and parameters, and we assume that the system of parameters β give a definite transformation T_β of the family, we have

$$f_i(f_1(x, \bar{a}) \ldots f_n\,(x, \bar{a}), \beta_1 \ldots \beta_r) = \lim_{a=\beta} f_i\,(f_1\,(x, \bar{a}) \ldots f_n\,(x, \bar{a}), a_1 \ldots a_r)$$
$$= \lim_{a=b} f_i\,(x_1 \ldots x_n,\, a_1 \ldots a_r) = f_i\,(x_1 \ldots x_n,\, b_1 \ldots b_r) \quad (i = 1, 2 \ldots n),$$

which is equivalent to the symbolic equation

$$T_{\bar{a}}\, T_\beta = T_{\bar{a}} \lim_{a=\beta} T_a = \lim_{a=\beta} T_{\bar{a}}\, T_a = \lim_{a=b} T_a = T_b.$$

Consequently, the composition of two arbitrary transformations $T_{\bar{a}}$ and T_β of the family is equivalent to a transformation T_b of this family; that is to say, the family of transformations T_a forms a group. The transformation T_b, however, may not be a transformation of the group that can be generated by an infinitesimal transformation of this group. Thus, every transformation of a group with continuous parameters is not necessarily generated by an infinitesimal transformation of the group.